Decay of correlations for Hénon maps

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Abstract

We show, for a class of automorphisms of $\mathbb{C}^k$, that their equilibrium measures are exponentially mixing. In particular, this holds for (generalized) Hénon maps in $\mathbb{C}^2$.

1 Introduction

Let f be a polynomial automorphism of $\mathbb{C}^k$. We also write f for its extension as a birational self-map of $\mathbb{P}^k$. When the exceptional sets $I_\pm$ of $f^\pm$ are not empty and satisfy $I_+ \cap I_- = \emptyset$, we say that f is a *regular automorphism* in the sense of Sibony. The reader may find a description of these maps in the survey of Sibony [22]. See also Friedland-Milnor [16], Bedford-Lyubich-Smillie [2, 1] and Fornæss-Sibony [14]. We recall here some facts.

The exceptional sets $I_\pm$ are contained in the hyperplane at infinity L_∞. There exists an integer s such that $\dim I_+ = k - 1 - s$ and $\dim I_- = s - 1$. We have $f(L_\infty \setminus I_+) = I_-$ and $f^{-1}(L_\infty \setminus I_-) = I_+$. Moreover, I_- is attractive for f and I_+ is attractive for f^{-1}. If d_+ and d_- are the algebraic degrees of f and f^{-1} respectively, then $d_+^s = d_-^{k-s} > 1$. We have $d_+ = d_-$ when $k = 2s$. In the dimension $k = 2$, f is a generalized Hénon automorphism which are the only dynamically interesting polynomial automorphisms of $\mathbb{C}^2$.

Sibony constructed for such a map an invariant probability measure μ (called *Green measure* or *equilibrium measure*) as an exterior product of positive closed $(1,1)$-currents:

$$\mu = T_+^s \wedge T_-^{k-s}$$

where $T_\pm$ are *Green currents of bidegree* $(1,1)$ and of mass 1 associated to $f^{\pm 1}$. They have local continuous potentials in $\mathbb{P}^k \setminus I_\pm$ and satisfy $f^*(T_+) = d_+ T_+$,

$f_*(T_-) = d_-T_-$. The current T_+^s (resp. T_-^{k-s}) is supported in the boundary of the *filled Julia set* $\mathcal{K}_+$ (resp. $\mathcal{K}_-$); it is called *Green current of bidegree* (s,s) (resp. $(k-s,k-s)$) associated to f (resp. to f^{-1}).

Recall that $\mathcal{K}_+$ (resp. $\mathcal{K}_-$) is the set of points $z \in \mathbb{C}^k$ such that the orbit $(f^n(z))_{n\in\mathbb{N}}$ (resp. $(f^{-n}(z))_{n\in\mathbb{N}}$) is bounded in $\mathbb{C}^k$. We have $\overline{\mathcal{K}}_\pm \cap L_\infty = I_\pm$. The open set $\mathbb{P}^k\setminus\overline{\mathcal{K}}_+$ (resp. $\mathbb{P}^k\setminus\overline{\mathcal{K}}_-$) is the immediate bassin of I_- for f (resp. I_+ for f^{-1}). The measure μ is supported in the boundary of the compact set $\mathcal{K} := \mathcal{K}_+ \cap \mathcal{K}_-$.

It was recently proved in [11, 19] that μ is mixing. This generalizes results of Bedford-Smillie [2] and Sibony [22]. The proofs follow the same approach and use the property that T_+^s and T_-^{k-s} are extremal currents. In this paper, we use another method to show that μ is mixing and the speed of mixing is exponential when $k = 2s$. We keep the above notation.

Theorem 1.1 *Let f be as above and assume that $k = 2s$. Then, there exists a constant $c > 0$ such that*

$$\left|\int (\varphi\circ f^n)\psi \mathrm{d}\mu - \left(\int \varphi \mathrm{d}\mu\right)\left(\int \psi \mathrm{d}\mu\right)\right| \leq c d_+^{-n/2}\|\varphi\|_{\mathcal{C}^2}\|\psi\|_{\mathcal{C}^2}$$

for all $n \geq 0$ and all real-valued $\mathcal{C}^2$ functions φ and ψ in $\mathbb{C}^k$.

Of course, this result holds for polynomial automorphisms of positive entropy in $\mathbb{C}^2$, in particular, for Hénon maps. We can apply it for any real Hénon map of degree d in $\mathbb{R}^2$ which admits an invariant probability measure μ of entropy $\log d$.

In [1], Bedford-Lyubich-Smillie proved for complex Hénon maps that the equilibrium measure is Bernoulli. This is the strongest mixing in the sense of measures. However, it does not imply the decay of correlations in our sense.

Observe also that in Theorem 1.1 we cannot replace $\|\varphi\|_{\mathcal{C}^2}$ or $\|\psi\|_{\mathcal{C}^2}$ by $\|\varphi\|_{\mathrm{L}^\infty}$ or $\|\psi\|_{\mathrm{L}^\infty}$ since, in general, $\psi\circ f^{-n}$ and $\varphi\circ f^n$ do not converge in $\mathrm{L}^1(\mu)$ to a constant.

We think that Theorem 1.1 should be true for every regular automorphism, for Hénon-like maps in dimension 2 [12, 6] and for larger classes of birational maps of $\mathbb{P}^k$ considered in [11]. For a general regular automorphism, in order to apply our approach, one needs to analyse the indeterminacy sets of some automorphisms close to the regular polynomial maps in the sense of [8] (see Lemma 3.2 below).

Note that the exponential decay of correlations has been proved for some polynomial-like maps and for meromorphic maps of large topological degree in [15, 7, 9]. In these cases, by the classical Gordin-Liverani theorem, our estimates imply the central limit theorem for bounded quasi-p.s.h. observables.

In Sections 2 and 3, we give some properties of the Green currents and the equilibrium measure. The method of dd^c-resolution developed in [7, 9, 10, 11] will be applied to establish the necessary estimates (Propositions 2.1, 3.1). We then deduce in Section 4 the mixing and the speed of mixing.

2 Convergence toward the Green current

Let us recall two properties of currents on $\mathbb{P}^k$ that will be used later on. Since $\mathbb{P}^k$ is homogeneous, every positive closed current S on $\mathbb{P}^k$ can be regularized on every neighbourhood U of $\mathrm{supp}(S)$. If T is a positive closed $(1,1)$-current with local continuous potentials in a neighbourhood of $\overline{U}$, then the positive closed current $T^m \wedge S$ is well defined and depends continuously on S. We refer to [13, 21, 3, 5, 22] for the basics of the theory of currents.

Now, consider a regular automorphism f on $\mathbb{C}^k$ as in Section 1. We do not suppose that $k = 2s$. Fix neighbourhoods U_i of $\overline{\mathcal{K}}_+$ and V_i of $\overline{\mathcal{K}}_-$ such that $f^{-1}(U_i) \Subset U_i$, $U_1 \Subset U_2$, $f(V_i) \Subset V_i$, $V_1 \Subset V_2$ and $U_2 \cap V_2 \Subset \mathbb{C}^k$. Observe that $\mathcal{K}_+ \cap \mathcal{K}_- \subset U_1 \cap V_1$.

Let Ω be a real $(k-s+1, k-s+1)$-current with support in $\overline{V}_1$. Assume that there exists a positive closed $(k-s+1, k-s+1)$-current Ω' supported in $\overline{V}_1$ such that $-\Omega' \leq \Omega \leq \Omega'$. Define the norm $\|\Omega\|_*$ of Ω as

$$\|\Omega\|_* := \min\{\|\Omega'\|,\ \Omega' \text{ as above}\}$$

where $\|\Omega'\| = \langle \Omega', \omega^{s-1} \rangle$ is the mass of Ω'. Here ω denotes the Fubini-Study form on $\mathbb{P}^k$ normalized by $\int \omega^k = 1$.

Keeping the above notation, the main result of this section is the following proposition.

Proposition 2.1 *Let R be a positive closed (s,s)-current of mass 1 supported in U_1 and smooth on $\mathbb{C}^k$. Let Φ be a real smooth $(k-s, k-s)$-form with compact support in $V_1 \cap \mathbb{C}^k$. Assume that $dd^c\Phi \geq 0$ in U_2. Then, there*

exist constants $c > 0$ independent of R, Φ, and $c_R > 0$ independent of Φ such that

$$\langle d_+^{-sn} f^{n*}(R) - T_+^s, \Phi\rangle \leq c d_+^{-n} \|\mathrm{dd^c}\Phi\|_*$$

and

$$\left|\langle d_+^{-sn} f^{n*}(R) - T_+^s, \Phi\rangle\right| \leq c_R d_+^{-n} \|\mathrm{dd^c}\Phi\|_*$$

for every $n \geq 0$. In particular, $d_+^{-sn} f^{n*}(R) \to T_+^s$ as $n \to \infty$.

The current $f^{n*}(R)$ is well defined since f^{-n} is holomorphic in U_1. We have

$$\begin{aligned} \langle d_+^{-sn} f^{n*}(R) - T_+^s, \Phi\rangle &= d_+^{-sn}\langle f^{n*}(R - T_+^s), \Phi\rangle \\ &= d_+^{-sn}\langle R - T_+^s, (f^n)_*\Phi\rangle. \end{aligned} \tag{1}$$

Since the currents R and T_+^s have the same mass 1, they are cohomologous. On $\mathbb{P}^k$, $R - T_+^s$ is $\mathrm{dd^c}$-exact. Hence, the last term in (1) does not change if we subtract a $\mathrm{dd^c}$-closed form from $(f^n)_*\Phi$. We will use the following lemma applied to $\mathrm{dd^c}(f^n)_*\Phi$.

Lemma 2.2 *Let Ω be a real smooth form of bidegree $(k-s+1, k-s+1)$ supported in $\overline{V}_1$ such that $\Omega \geq 0$ on U_2 and $\|\Omega\|_* \leq 1$. Assume that Ω is $\mathrm{dd^c}$-exact. Then there exist $c > 0$ independent of Ω and a real continuous $(k-s, k-s)$-form Ψ such that $\mathrm{dd^c}\Psi = \Omega$, $\|\Psi\| \leq c$, $\Psi \leq 0$ on U_1 and $\Psi \geq -c\omega^{k-s}$ on $\mathbb{P}^k \setminus V_2$.*

Proof. By Hodge theory [18], we have

$$H^{k,k}(\mathbb{P}^k \times \mathbb{P}^k, \mathbb{C}) \simeq \sum_{p+p'=k} H^{p,p}(\mathbb{P}^k, \mathbb{C}) \otimes H^{p',p'}(\mathbb{P}^k, \mathbb{C}).$$

Hence, if Δ is the diagonal of $\mathbb{P}^k \times \mathbb{P}^k$, there exists a smooth real (k,k)-form $\alpha(x,y)$ on $\mathbb{P}^k \times \mathbb{P}^k$, cohomologous to $[\Delta]$, with $\mathrm{d}_x\alpha = \mathrm{d}_y\alpha = 0$. Since $\mathbb{P}^k \times \mathbb{P}^k$ is homogeneous, following [4, Prop. 6.2.3] (see also [17, 10, 11]), one can construct a negative $(k-1, k-1)$-form $K(x,y)$ on $\mathbb{P}^k \times \mathbb{P}^k$, smooth outside Δ, such that $\mathrm{dd^c}K = [\Delta] - \alpha$ and $|K(x,y)| \leq A|x-y|^{1-2k}$ for some constant $A > 0$. Here $|x-y|$ denotes the distance between x and y.

Define

$$\Psi'(x) := \int_y K(x,y) \wedge \Omega(y).$$

From the estimate, one check easily that Ψ' is continuous and $\|\Psi'\| \leq c'$, $\Psi' \leq c'\omega^{k-s}$ on U_1, $\Psi' \geq -c'\omega^{k-s}$ on $\mathbb{P}^k \setminus V_2$, where $c' > 0$ is independent of Ω. Define $\Psi := \Psi' - c'\omega^{k-s}$. We obtain $\|\Psi\| \leq 2c'$, $\Psi \leq 0$ on U_1 and $\Psi \geq -2c'\omega^{k-s}$ on $\mathbb{P}^k \setminus V_2$. We only have to verify that $\mathrm{dd^c}\Psi' = \Omega$.

Since Ω is $\mathrm{dd^c}$-exact and $\mathrm{d}_x\alpha = \mathrm{d}_y\alpha = 0$, we have

$$\begin{aligned}
\mathrm{dd^c}\Psi'(x) &:= \int_y (\mathrm{dd^c})_x K(x,y) \wedge \Omega(y) = \int_y \mathrm{dd^c} K(x,y) \wedge \Omega(y) \\
&= \int_y ([\Delta] - \alpha) \wedge \Omega(y) = \Omega(x) - \int_y \alpha \wedge \Omega(y) \\
&= \Omega(x).
\end{aligned}$$

Hence, $\mathrm{dd^c}\Psi = \mathrm{dd^c}\Psi' = \Omega$.

□

Proof of Proposition 2.1. We can assume that $\|\mathrm{dd^c}\Phi\|_* = 1$. The constants c and c_i below are independent of Φ and R. Define $\Omega := \mathrm{dd^c}\Phi$. By hypotheses, there exists a positive closed current Ω' of mass 1 supported in $\overline{V}_1$ such that $-\Omega' \leq \Omega \leq \Omega'$. Define $\Omega_n := \mathrm{dd^c}(f^n)_*\Phi = (f^n)_*\Omega$ and $\Omega'_n := (f^n)_*\Omega'$. These currents have supports in $\overline{V}_1$ since $f^n(V_1) \Subset V_1$. We also have $-\Omega'_n \leq \Omega_n \leq \Omega'_n$ and $\Omega_n \geq 0$ on U_2 since $f^{-n}(U_2) \Subset U_2$. A simple calculus on cohomology gives $\|\Omega'_n\| = d_+^{(s-1)n}\|\Omega'\| = d_+^{(s-1)n}$. Lemma 2.2 implies the existence of Ψ_n cohomologous to $(f^n)_*\Phi$ such that $\Psi_n \leq 0$ on U_1, $\Psi_n \geq -cd_+^{(s-1)n}\omega^{k-s}$ on $\mathbb{P}^k \setminus V_2$ and $\|\Psi_n\| \leq cd_+^{(s-1)n}$. In particular, $\Psi_n \leq 0$ on $\mathrm{supp}(R)$. Therefore, we deduce from (1) that

$$\langle d_+^{-sn} f^{n*}(R) - T_+^s, \Phi\rangle = d_+^{-sn}\langle R - T_+^s, \Psi_n\rangle \leq -d_+^{-sn}\langle T_+^s, \Psi_n\rangle. \tag{2}$$

We have to bound $-\langle T_+^s, \Psi_n\rangle$. Since T_+ has continuous potentials in $\mathbb{P}^k \setminus I_+$, we can write $T_+ = \omega + \mathrm{dd^c}u$ with $u \leq 0$ and u continuous on $\mathbb{P}^k \setminus I_+$. One has

$$\begin{aligned}
|\langle T_+^s, \Psi_n\rangle| &= |\langle \omega \wedge T_+^{s-1} + \mathrm{dd^c}(uT_+^{s-1}), \Psi_n\rangle| \\
&\leq |\langle T_+^{s-1}, \omega \wedge \Psi_n\rangle| + |\langle uT_+^{s-1}, \mathrm{dd^c}\Psi_n\rangle| \\
&\leq |\langle T_+^{s-1}, \omega \wedge \Psi_n\rangle| - \langle uT_+^{s-1}, \Omega'_n\rangle. \qquad (3)
\end{aligned}$$

Since Ω'_n has support in $\overline{V}_1$ where u is bounded, the second term in the last line of (3) is dominated by $c_1\langle T_+^{s-1}, \Omega'_n\rangle$. The integral $\langle T_+^{s-1}, \Omega'_n\rangle$ is cohomological; it is equal to $\|\Omega'_n\|$. Hence, $-\langle uT_+^{s-1}, \Omega'_n\rangle \leq c_1 d_+^{(s-1)n}$.

For the first term in the last line of (3), we write $T_+^{s-1} = \omega \wedge T_+^{s-2} + dd^c(uT_+^{s-2})$. Using expansions as in (3) and an induction argument, we get $|\langle T_+^{s-1}, \omega \wedge \Psi_n\rangle| \leq c_2 d_+^{(s-1)n}$. At the last step of the induction, we use the inequality $\|\Psi_n\| \leq c d_+^{(s-1)n}$. Hence, the first part of Proposition 2.1 follows.

For the second part, it is sufficient to prove that $|\langle R, \Psi_n\rangle| \leq c'_R d_+^{(s-1)n}$ with c'_R independent of Φ. This follows directly from the smoothness of R on $\mathbb{C}^k$ and the properties that $\|\Psi_n\| \leq c d_+^{(s-1)n}$ and $-c d_+^{(s-1)n}\omega^s \leq \Psi_n \leq 0$ on the neighbourhood $U_1 \setminus V_2$ of the singularities of R.

Now, we show that $d_+^{-sn} f^{n*}(R) \to T_+^s$ on $\mathbb{C}^k$. Consider a real smooth test $(k-s, k-s)$-form Φ with compact support in $\mathbb{C}^k$. We want to prove that $\langle d^{-sn} f^{n*}(R) - T_+^s, \Phi\rangle \to 0$. Observe that $\mathbb{P}^k \setminus I_+$ is a union of compact algebraic sets of dimension s. Hence, we can construct a positive closed $(k-s, k-s)$-form Θ supported in $\mathbb{P}^k \setminus I_+$ and strictly positive on $\mathrm{supp}(\Phi)$. Since

$$\langle d_+^{-sn} f^{n*}(R) - T_+^s, \Phi\rangle = \langle d_+^{-s(n-m)} f^{(n-m)*}(R) - T_+^s, d_+^{-sm}(f^m)_*\Phi\rangle,$$

replacing Φ and Θ by $d_+^{-sm}(f^m)_*\Phi$ and $(f^m)_*\Theta$, m big enough, one can assume that $\mathrm{supp}(\Theta) \subset V_1$.

Consider a smooth function χ with compact support in $\mathbb{C}^k$ and strictly p.s.h. on neighbourhood of $U_2 \cap V_2$. Write $\Phi = (\Phi + A\chi\Theta) - A\chi\Theta$ with $A > 0$ big enough, so that $dd^c(\Phi + A\chi\Theta)$ and $dd^c(A\chi\Theta)$ are positive on U_2. Hence, it is sufficient to consider the case where $dd^c\Phi \geq 0$ on U_2. The second part of the proposition implies that $\langle d^{-sn} f^{n*}(R) - T_+^s, \Phi\rangle \to 0$.

□

3 Convergence toward the Green measure

In this section, we will apply Proposition 2.1 to the automorphism F constructed in Lemma 3.2 below. Our main result is the following proposition.

Proposition 3.1 *Let f be as above with $k = 2s$. Let φ be a smooth function on $\mathbb{P}^k$ and p.s.h. on $U_2 \cap V_2$. Let R (resp. S) be a positive closed (s,s)-current of mass 1 with support in U_1 (resp. in V_1) and smooth on $\mathbb{C}^k$. Then, there exist constants $c > 0$ independent of φ, R, S, and $c_{R,S} > 0$ independent of φ*

such that
$$\left\langle d_+^{-2sn} f^{n*}(R)\wedge (f^n)_*(S)-\mu,\varphi\right\rangle \leq c d_+^{-n}\|\varphi\|_{\mathcal{C}^2}$$
and
$$\left|\left\langle d_+^{-2sn} f^{n*}(R)\wedge (f^n)_*(S)-\mu,\varphi\right\rangle\right| \leq c_{R,S} d_+^{-n}\|\varphi\|_{\mathcal{C}^2}$$
for every $n\geq 0$. In particular, $d_+^{-2sn} f^{n*}(R)\wedge (f^n)_*(S)\to\mu$ as $n\to\infty$.

We will use z, w and (z,w) for the canonical coordinates of complex spaces $\mathbb{C}^k$ and $\mathbb{C}^k\times\mathbb{C}^k$. Consider also the canonical inclusions of $\mathbb{C}^k$ and $\mathbb{C}^k\times\mathbb{C}^k$ in $\mathbb{P}^k$ and $\mathbb{P}^{2k}$. We write $[z:t]$, $[w:t]$ or $[z:w:t]$ for the homogeneous coordinates of projective spaces. The hyperplanes at infinity are defined by $t=0$. If $g:\mathbb{C}^k\to\mathbb{C}^k$ is a polynomial automorphism, we write g_h (resp. g_h^{-1}) for the homogeneous part of maximal degree of g (resp. of g^{-1}). They are self-maps of $\mathbb{C}^k$, not invertible in general. In the sequence, we always assume that $k=2s$.

Lemma 3.2 *Let F be the automophism of $\mathbb{C}^k\times\mathbb{C}^k$ defined by $F(z,w):=(f(z),f^{-1}(w))$. Then F is regular. The indeterminacy sets $I_\pm^F$ of $F^\pm$ are defined by*
$$I_\pm^F := \left\{[z:w:0],\ f_h^{\pm 1}(z)=0,\ f_h^{\mp 1}(w)=0\right\}.$$
Moreover, if $\Delta:=\{z=w\}$ is the diagonal of $\mathbb{C}^k\times\mathbb{C}^k$, then $I_\pm^F$ do not intersect $\overline{\Delta}$. In particular, $F(\overline{\Delta})\cap\{t=0\}\subset I_-^F$.

Proof. Since $k=2s$, we have $d_+=d_-$ and $F_h^{\pm 1}(z,w)=(f_h^{\pm 1}(z),f_h^{\mp 1}(w))$. It follows that
$$I_\pm^F=\left\{[z:w:0],\ F_h^{\pm 1}(z,w)=0\right\}=\left\{[z:w:0],\ f_h^{\pm 1}(z)=f_h^{\mp 1}(w)=0\right\}.$$
We also have
$$I_\pm := \left\{[z:0],\ f_h^{\pm 1}(z)=0\right\}$$
and since f is regular,
$$\left\{z\in\mathbb{C}^k,\ f_h(z)=f_h^{-1}(z)=0\right\}=\{0\}.$$
This implies that $I_+^F\cap I_-^F=\emptyset$. Hence, F is regular. We also have
$$I_\pm^F\cap\overline{\Delta}=\left\{[z:z:0],\ f_h(z)=f_h^{-1}(z)=0\right\}=\emptyset.$$
□

Lemma 3.3 *Under the notation of Lemma 3.2, the Green current of bidegree $(2s, 2s)$ of F is equal to $T_+^s \otimes T_-^s$.*

Proof. Let R and S be as in Proposition 3.1. Replacing R and S by $d_+^{-s} f^*(R)$ and $d_+^{-s} f_*(S)$, we get $\mathrm{supp}(R) \cap \{t = 0\} \subset I_+$ and $\mathrm{supp}(S) \cap \{t = 0\} \subset I_-$.

Consider the current $R \otimes S$ in $\mathbb{C}^k \times \mathbb{C}^k$ and in $\mathbb{P}^{2k}$. Lemma 3.2 implies $\overline{\mathrm{supp}(R \otimes S)} \cap \{t = 0\} \subset I_+^F$. Since $\dim I_+^F = 2s - 1$, the trivial extension of $R \otimes S$ in $\mathbb{P}^{2k}$ (that we denote also by $R \otimes S$) is a positive closed current [20]. One can check that the mass of $R \otimes S$ is equal to 1. Proposition 2.1 applied to F implies that $d_+^{-2sn} F^{n*}(R \otimes S)$ converge to the Green current of bidegree $(2s, 2s)$ of F. On the other hand, we have

$$d_+^{-2sn} F^{n*}(R \otimes S) = d_+^{-2sn} f^{n*}(R) \otimes (f^n)_*(S) \to T_+^s \otimes T_-^s$$

in $\mathbb{C}^k \times \mathbb{C}^k$. Hence, $T_+^s \otimes T_-^s$ is the Green $(2s, 2s)$-current of F.

$\square$

Proof of Proposition 3.1. We can assume that φ has compact support in $\mathbb{C}^k$ and $\|\varphi\|_{\mathcal{C}^2} = 1$. As in Lemma 3.3, we can assume that the current $R \otimes S$ in $\mathbb{P}^{2k}$ satisfies $\mathrm{supp}(R \otimes S) \cap \{t = 0\} \subset I_+^F$.

Define $\widehat{\varphi}(z, w) := \varphi(z)$. Since $T_\pm$ are invariant and have continuous potentials out of $I_\pm$, we can write

$$\begin{aligned} &\left\langle d_+^{-2sn} f^{n*}(R) \wedge (f^n)_*(S) - \mu, \varphi \right\rangle \\ &\quad = \left\langle d_+^{-2sn} f^{n*}(R) \otimes (f^n)_*(S) - T_+^s \otimes T_-^s, \widehat{\varphi}[\Delta] \right\rangle. \end{aligned}$$

Using a regularization of $[\Delta]$, one may find a smooth current Θ of mass 1 supported in a small neighbourhood $\mathcal{W}$ of $\overline{\Delta}$ such that

$$\begin{aligned} &\Big| \left\langle d_+^{-2sn} f^{n*}(R) \otimes (f^n)_*(S) - T_+^s \otimes T_-^s, \widehat{\varphi}[\Delta] \right\rangle - \\ &\quad - \left\langle d_+^{-2sn} f^{n*}(R) \otimes (f^n)_*(S) - T_+^s \otimes T_-^s, \widehat{\varphi}\Theta \right\rangle \Big| \leq d_+^{-n}. \end{aligned}$$

The current Θ depends on n and $\mathcal{W} \cap I_+^F = \emptyset$ (see Lemma 3.2).

We have to estimate

$$\left\langle d_+^{-2sn} f^{n*}(R) \otimes (f^n)_*(S) - T_+^s \otimes T_-^s, \widehat{\varphi}\Theta \right\rangle.$$

Fix an integer $m > 0$ big enough. Write

$$\begin{aligned} &\left\langle d_+^{-2sn} f^{n*}(R) \otimes (f^n)_*(S) - T_+^s \otimes T_-^s, \widehat{\varphi}\Theta \right\rangle \\ &\quad = \left\langle d_+^{-2sn} F^{n*}(R \otimes S) - d_+^{-2sm} F^{m*}(T_+^s \otimes T_-^s), \widehat{\varphi}\Theta \right\rangle \\ &\quad = \left\langle d_+^{-2s(n-m)} F^{(n-m)*}(R \otimes S) - T_+^s \otimes T_-^s, d_+^{-2sm} (F^m)_*(\widehat{\varphi}\Theta) \right\rangle \\ &\quad =: \left\langle d_+^{-2s(n-2m)} F^{(n-2m)*}(T) - T_+^s \otimes T_-^s, \Phi \right\rangle \end{aligned}$$

where $T := d_+^{-2sm} F^{m*}(R \otimes S)$ and $\Phi := d_+^{-2sm}(F^m)_*(\widehat{\varphi}\Theta)$.

Hence, T has support in a small neighbourhood $\mathcal{U}$ of the filled Julia set $\mathcal{K}_+^F = \mathcal{K}_+ \times \mathcal{K}_-$ of F and Φ is a smooth form with support in a small neighbourhood $\mathcal{V}$ of $\mathcal{K}_-^F = \mathcal{K}_- \times \mathcal{K}_+$. Moreover, since m is big and φ is p.s.h. on $U_2 \cap V_2$, $\mathrm{dd^c}\Phi \geq 0$ in a neighbourhood $\mathcal{U}' \Supset \mathcal{U}$ of $\mathcal{K}_+^F$. Putting $\widehat{\omega}(z, w) := \omega(z)$, we have $-\widehat{\omega} \leq \mathrm{dd^c}\widehat{\varphi} \leq \widehat{\omega}$ since $\|\varphi\|_{\mathcal{C}^2} = 1$. It follows that

$$-d_+^{-2sm}(F^m)_*(\widehat{\omega} \wedge \Theta) \leq \mathrm{dd^c}\Phi \leq d_+^{-2sm}(F^m)_*(\widehat{\omega} \wedge \Theta).$$

The choice of $\mathcal{W}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$ and m does not depend on φ and n. Lemma 3.3 and Proposition 2.1 applied to F, T and Φ imply

$$\left\langle d_+^{-2s(n-2m)} F^{(n-2m)*}(T) - T_+^s \otimes T_-^s, \Phi \right\rangle \leq c' d_+^{-n}$$

and

$$\left| \left\langle d_+^{-2s(n-2m)} F^{(n-2m)*}(T) - T_+^s \otimes T_-^s, \Phi \right\rangle \right| \leq c'_T d_+^{-n}.$$

The desired inequalities of the proposition follow. Since every smooth test function on $\mathbb{P}^k$ can be written as a difference of smooth functions p.s.h. on $U_2 \cap V_2$, these inequalities imply that $d_+^{-2sn} f^{n*}(R) \wedge (f^n)_*(S) \to \mu$. $\square$

Corollary 3.4 *The Green measure of F is equal to $\mu \otimes \mu$.*

Proof. Let R and S be as in Proposition 3.1 such that $\mathrm{supp}(R \otimes S) \cap \{t = 0\} \subset I_+^F$ and $\mathrm{supp}(S \otimes R) \cap \{t = 0\} \subset I_-^F$. Proposition 3.1 implies that the Green measure of F is equal to

$$\begin{aligned}
&\lim d_+^{-4sn} F^{n*}(R \otimes S) \wedge (F^n)_*(S \otimes R) \\
&\quad = \lim d_+^{-4sn} \left[f^{n*}(R) \otimes (f^n)_*(S)\right] \wedge \left[(f^n)_*(S) \otimes f^{n*}(R)\right] \\
&\quad = \lim \left[d_+^{-2sn} f^{n*}(R) \wedge (f^n)_*(S)\right] \otimes \left[d_+^{-2sn} f^{n*}(R) \wedge (f^n)_*(S)\right] \\
&\quad = \mu \otimes \mu.
\end{aligned}$$

$\square$

4 Speed of mixing

In this section, we give the proof of Theorem 1.1. Fix a domain D in $\mathbb{C}^k$ containing $\mathcal{K} := \mathcal{K}_+ \cap \mathcal{K}_-$. Observe that φ and ψ can be written as differences

of smooth functions strictly p.s.h. on a neighbourhood of $\overline{D}$. Hence, we can assume that $dd^c\varphi \geq \omega$ on D, $dd^c\psi \geq \omega$ on D and $\|\varphi\|_{\mathcal{C}^2} \leq M$, $\|\psi\|_{\mathcal{C}^2} \leq M$ for some fixed constant $M > 0$. The constants c, A, c' below do not depend on φ and ψ.

It is sufficient to prove Theorem 1.1 for n even. So we have to prove that

$$\left|\left\langle \mu, (\varphi \circ f^n)(\psi \circ f^{-n})\right\rangle - \langle \mu, \varphi\rangle\langle \mu, \psi\rangle\right| \leq c d_+^{-n}. \tag{4}$$

Observe that, since μ is invariant, the left hand side of (4) does not change if we add to φ or to ψ a constant. Consequently, one only need to check for a constant A that

$$\left\langle \mu, (\varphi \circ f^n + A)(\psi \circ f^{-n} + A)\right\rangle - \langle \mu, \varphi + A\rangle\langle \mu, \psi + A\rangle \leq c d_+^{-n} \tag{5}$$

and

$$\left\langle \mu, (\varphi \circ f^n - A)(-\psi \circ f^{-n} + A)\right\rangle - \langle \mu, \varphi - A\rangle\langle \mu, -\psi + A\rangle \leq c d_+^{-n}. \tag{6}$$

We choose $A > 0$ big enough so that $\phi(z, w) := (\varphi(z) + A)(\psi(w) + A)$ and $\phi'(z, w) := (\varphi(z) - A)(-\psi(w) + A)$ are p.s.h. on $D \times D$. This allows to apply Proposition 3.1 to the automorphism F and to the test functions ϕ, ϕ'. We will check (5). The estimate (6) can be proved in the same way.

Define $T_1 := T_+^s \otimes T_-^s$. Since $F^*(T_1) = d_+^{2s} T_1$ and $T_\pm$ have continuous potentials in $\mathbb{C}^k$, we get the following identities

$$\begin{aligned}
\left\langle \mu, (\varphi \circ f^n + A)(\psi \circ f^{-n} + A)\right\rangle &= \left\langle T_+^s \wedge T_-^s, (\varphi \circ f^n + A)(\psi \circ f^{-n} + A)\right\rangle \\
&= \langle T_1 \wedge [\Delta], \phi \circ F^n\rangle \\
&= \left\langle d_+^{-4sn+2sm} F^{(2n-m)*}(T_1) \wedge [\Delta], \phi \circ F^n\right\rangle \\
&= \left\langle d_+^{-4sn+2sm} F^{(n-m)*}(T_1) \wedge (F^n)_*[\Delta], \phi\right\rangle \\
&=: \left\langle d_+^{-4sn+4sm} F^{(n-m)*}(T_1) \wedge (F^{n-m})_* T_2, \phi\right\rangle
\end{aligned}$$

where $T_2 := d^{-2sm}(F^m)_*[\Delta]$ and m is a fixed, but sufficiently large integer. By Lemma 3.2, T_2 has support in a small neighbouhood $\mathcal{V}$ of $\mathcal{K}_-^F$.

Using a regularization of currents, we may find smooth currents T_1' and T_2' with support in small neighbourhoods $\mathcal{U}$ of $\mathcal{K}_+^F$ and $\mathcal{V}$ of $\mathcal{K}_-^F$ respectively, so that

$$\begin{aligned}
&\left\langle d_+^{-4sn+4sm} F^{(n-m)*}(T_1) \wedge (F^{n-m})_* T_2, \phi\right\rangle - \\
&\quad - \left\langle d_+^{-4sn+4sm} F^{(n-m)*}(T_1') \wedge (F^{n-m})_* T_2', \phi\right\rangle \leq d_+^{-n}.
\end{aligned}$$

The currents T'_1 and T'_2 depend on n. The choice of m, $\mathcal{U}$, $\mathcal{V}$ depends only on D and f with $\mathcal{U} \cap \mathcal{V} \Subset D \times D$.

Since $\langle \mu, \varphi + A \rangle \langle \mu, \psi + A \rangle = \langle \mu \otimes \mu, \phi \rangle$, we only have to check that

$$\left\langle d_+^{-4sn+4sm} F^{(n-m)*}(T'_1) \wedge (F^{n-m})_* T'_2 - \mu \otimes \mu, \phi \right\rangle \leq c' d_+^{-n}.$$

This inequality follows directly from Corollary 3.4 and Proposition 3.1 applied to F and ϕ. Hence, the proof of Theorem 1.1 is complete.

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References

[1] Bedford E., Lyubich M., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$ IV: The measure of maximal entropy and laminar currents, *Invent. Math.*, **112** (1993), 77-125.

[2] Bedford E., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$ III: Ergodicity, exponents and entropy of the equilibrium measure, *Math. Ann.*, **294** (1992), 395-420.

[3] Bedford E., Taylor B.A., A new capacity for plurisubharmonic functions, *Acta Math.*, **149** (1982), 1-39.

[4] Bost J.B., Gillet H., Soulé C., Heights of projective varieties and positive Green forms, *J. Amer. Math. Soc.*, **7** (1994), no. 4, 903–1027.

[5] Demailly J.P., Monge-Ampère Operators, Lelong numbers and Intersection theory in Complex Analysis and Geometry, *Plemum Press* (1993), 115-193, *(V. Ancona and A. Silva editors).*

[6] Dinh T.C., Dujardin R., Sibony N., On the dynamics near infinity of some polynomial mappings in $\mathbb{C}^2$, *preprint*, 2004. arxiv.org/abs/math.DS/0407451.

[7] Dinh T.C., Sibony N., Dynamique des applications d'allure polynomiale, *Journal de Math. Pures et Appl.*, **82** (2003), 367-423.

[8] Dinh T.C., Sibony N., Dynamique des applications polynomiales semi-régulières, *Arkiv för Matematik*, **42** (2004), 61-85.

[9] Dinh T.C., Sibony N., Distribution des valeurs d'une suite de transformations méromorphes et applications, *preprint*, 2003. arxiv.org/abs/math.DS/0306095.

[10] Dinh T.C., Sibony N., Green currents for automorphisms of compact Kähler manifolds, *preprint*, 2003. arxiv.org/abs/math.DS/0311322.

[11] Dinh T.C., Sibony N., Dynamics of regular birational maps in $\mathbb{P}^k$, *J. Funct. Analysis*, to appear. arxiv.org/abs/math.DS/0406367.

[12] Dujardin R., Hénon-like mappings in $\mathbb{C}^2$, *Amer. J. Math.*, **126** (2004), 439-472.

[13] Federer H., Geometric Measures theory, NewYork, Springer Verlag, 1969.

[14] Fornæss J.E., Sibony N., Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Biebebach domains, *Duke Math. J.*, **65** (1992), 245-250.

[15] Fornæss J.E., Sibony N., Complex dynamics in higher dimension, *in Complex potential theory*, (Montréal, PQ, 1993), Nato ASI series Math. and Phys. Sci., vol. **C439**, Kluwer (1994), 131-186.

[16] Friedland S., Milnor J., Dynamical properties of plane polynomial automorphisms, *Ergodic Theory & Dynamical Systems*, **9** (1989), 67-99.

[17] Gillet H., Soulé C., Arithmetic intersection theory, *I.H.E.S. Publ. Math.*, **72** (1990), 93–174 (1991).

[18] Griffiths P., Harris J., Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.

[19] Guedj V., Courants extrémaux et dynamique complexe, *preprint*.

[20] Harvey R., Polking J., Extending analytic objects, *Comm. Pure Appl. Math.*, **28** (1975), 701-727.

[21] Lelong P., Fonctions plurisousharmoniques et formes différentielles positives, Dunod Paris, 1968.

[22] Sibony N., Dynamique des applications rationnelles de $\mathbb{P}^k$, *Panoramas et Synthèses*, **8** (1999), 97-185.

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